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ON DISCONTINUOUS MOTIONS IN SYSTEMS WITH UNILATERAL CONSTRAINTS†

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Mechanical systems with a finite number of degrees of freedom, subject to one or more unilateral geometric constraints, are considered. Apart from the main forms of motion—flying, supported motion and non-degenerate collisions—such systems also show more complex, discontinuous motions, including infinitely many impacts in any neighbourhood of the starting time. These motions are possible not only when no continuous motions exist, but also together with continuous motions [1]. It is proved that, in the case of ideal constraints, if the reactions of the constraints at the starting time are non-zero, there cannot be any discontinuous motion. In systems with dry friction there is yet another type of discontinuity, namely, tangential impact at zero approach velocity. Sufficient conditions for continuity of the motion are derived for this case also. The conditions are verified with examples that use the usual models of impact. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

The equations of motion of a mechanical system in Lagrangian form are

$$\dot{p}_i - \frac{\partial T}{\partial q_i} = Q_i, \quad p_i = \frac{\partial T}{\partial \dot{q}_i} \quad (i = 1, \ldots n) \tag{1.1}$$

where T is the kinetic energy, set up taking into account the existing ideal bilateral constraints and Q_i is the generalized force corresponding to the generalized coordinate q_i. Under the usual conditions on the functions Q_i and T (e.g. continuous differentiability in some domain), the Cauchy problem for system (1.1) is well posed and the solutions $\mathbf{q}(t)$ are twice differentiable. Consequently, the phase curves are continuous.

When the system is subject to a unilateral constraint $q_1 \geqslant 0$, Eqs (1.1) only hold inside the domain of possible motion $q_1 > 0$, while on the boundary $q_1 = 0$ there are two possible types of motion, described by different equations. In the first type—supported motion—the constraint remains active over a certain time interval, and the reaction of the constraint is added to the forces acting on the system

$$\dot{p}_i - \frac{\partial T}{\partial q_i} = Q_i + R_i \tag{1.2}$$

System (1.2) contains a redundant number of unknowns, and it is therefore necessary to establish rules for solving it. One usually proceeds as follows. First, relying on physical considerations, one specifies some friction law, so that $R_2, \ldots, R_n$ can be expressed in terms of R_1. One then eliminates the generalized accelerations $\ddot{q}_2, \ldots, \ddot{q}_n$ so that in the end (for the classical friction laws) a single linear equation remains in the two variables $\ddot{q}_1$ and R_1. To eliminate a redundant unknown quantity one uses the so-called complementarity condition [2]. The result is a system of the form

$$\ddot{q}_1 = A_0 + A_1 R_1 \tag{1.3}$$

$$\ddot{q}_1 \geqslant 0, \quad R_1 \geqslant 0, \quad \ddot{q}_1 R_1 \equiv 0 \tag{1.4}$$

The coefficients A_0 and A_1 in Eq. (1.3) generally depend on the phase variables and on time.

If the unilateral constraint is ideal, then $A_1 > 0$ and system (1.3), (1.4) has a unique solution. If there is dry (Coulomb) friction, A_1 may also take negative values; in that case, depending on the sign of A_0, Eq. (1.3) either has several solutions or has no solutions compatible with (1.4). These situations are conventionally known as the Painlevé paradoxes [3].

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The specific feature of impacts is the discontinuous nature of the phase curves $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$. Impacts take place at those times for which $q_1 = 0, \dot{q}_1 < 0$ and are described by the equations

$$p_i^+ - p_i^- = I_i(\mathbf{q}, \mathbf{q}) \quad (i = 1, ..., n) \tag{1.5}$$

where $\mathbf{I}$ is a vector-valued function describing the velocity jumps ($I_1 \geqslant 0, \dot{q}_1^+ \geqslant 0$), and the minus and plus superscripts correspond to the beginning and end of the impact.

The form of the function $\mathbf{I}(\mathbf{q}, \dot{\mathbf{q}})$ depends on the physical properties of the impacting solids; it is important that in any case the kinetic energy will not increase on impact.

We now define a solution of the equations of motion in a system with a unilateral constraint, taking into account the possibility of weakening the constraint, of a supported phase and of impacts.

Definition. A continuous vector-valued function $\mathbf{q}(t)$ will be called a solution of system (1.1), (1.2), (1.5), if the following conditions are satisfied:

1. at times when $q_1 > 0$, the function $\mathbf{q}(t)$ is twice differentiable and satisfies system (1.1);
2. at every instant of time for which $q_1 = 0$, one-sided derivatives $\dot{\mathbf{q}}(t \pm 0)$ exist and equalities (1.5) hold;
3. if $q_1 = 0, \dot{q}_1 = 0$ at some instant of time, then $\ddot{\mathbf{q}}(t)$ exists and conditions (1.4) are satisfied.

Remark. Moreau [4] imposes an additional condition on the solutions: the generalized velocity $\dot{\mathbf{q}}(t)$ is of bounded variation in any bounded time interval. We will show that this property of mechanical systems follows from energy considerations. Indeed, if the variation were unbounded, this would mean that the sum of normal components of the impact impulses were unbounded. Hence the quantity p_1 would also be unbounded and, consequently, the kinetic energy of the system also. Such behaviour of the system over a finite time interval would imply the existence of an external source of energy of infinitely large capacity.

The primary problem of the theory of systems with non-restoring constraints is to determine the conditions for the existence and uniqueness of solutions and to determine how they depend on the initial data and parameters. Up to the present, a variety of results have been obtained in various special cases. Most attention has been paid to the case of an ideal constraint (that is, $R_j = 0, R_j = 0$ $(j = 2, \ldots, n)$) in Eqs (1.2) and (1.5)) with impacts described in accordance with the Newton hypothesis by the formula

$$\dot{q}_1^+ = -e\dot{q}_1^-, \quad e \in (0,1] \tag{1.6}$$

with a constant coefficient of restitution e.

The existence of a solution with given initial data has been proved [5]. The solution may be non-differentiable in any neighbourhood of the starting time t_0, owing to the presence of impacts at $t = t_k$ $(k = 1, 2, \ldots)$, where the sequence $\{t_k\}$ decreases monotonically to t_0 [1]. In the example constructed in [1] such an infinite-impact solution exists, apart from the impact-free solution for which $q_1 \equiv 0$. In the billiard system considered in [6] there are no impact-free solutions, though at the starting time $q_1 = 0, \dot{q}_1 = 0$.

Sufficient conditions for uniqueness in a billiard-type system were obtained in [7, 8]: the billiard boundary either has strictly negative Gauss curvature or is a level curve of a real-analytic function.

The aim of this paper is to derive the conditions for the solution to be unique in systems with one or more (not necessarily ideal) constraints.

If the system has a continuously differentiable solution for certain initial data, that solution is unique if and only if there are no impact interactions at $t > t_0$.

2. THE CASE OF AN IDEAL UNILATERAL CONSTRAINT

We will first assume that the system is subject to an ideal unilateral constraint, which corresponds to contact of bodies with convex smooth surfaces without edges (in which case q_1 is the distance between the bodies). The kinetic energy is a quadratic form in the generalized velocities (if there are linear terms, the terms generated by them in the equations of motion (1.1) may be classed as generalized forces). The coefficients of this form, as well as the functions $Q_i(t, \mathbf{q}, \dot{\mathbf{q}})$, are assumed to be continuously differentiable in the domain of possible motions.

One of the most popular models of the theory of vibrating-impact systems is a rigid body moving on a massive base which itself is moving in accordance with a known law. In that case Eqs (1.1) and (1.2) include the kinetic energy of relative motion, and the generalized forces include the forces of inertia.

By virtue of the equations $R_2 = \ldots = R_n \equiv 0, I_2 = \ldots = I_n \equiv 0$ one can derive several simple relations describing the impact. It follows from (1.5) that

$$p_j^+ = p_j^- \quad (j = 2, \ldots, n) \tag{2.1}$$

Let us express the kinetic energy in terms of the generalized velocity $\dot{q}_1$ and generalized momenta p_j. Fairly easy algebraic arguments show (see [9]) that it splits into a sum of two positive-definite quadratic forms, the first of which depends only on $\dot{q}_1$ and the second only on p_j (the coefficients may depend on the coordinates)

$$T = T_0(\dot{q}_1) + T^*(p_2, \ldots, p_n) \tag{2.2}$$

In view of (2.1), the second term in (2.2) remains unchanged on impact. Consequently, T_0 is a non-increasing quantity, or, equivalently

$$|\dot{q}_1^+| \leqslant |\dot{q}_1^-| \tag{2.3}$$

To obtain a closed system of equations for the impact we must add to (2.1) a relation of the form

$$\dot{q}_1^+ = f(\mathbf{q}, \dot{q}_1^-, p_2, \ldots, p_n) \tag{2.4}$$

which is compatible with (2.3) (a special case of such a law is (1.6)).

Let us assume that at the starting time $q_1(t_0) > 0$ or $q_1(t_0) = 0$, $q_1(t_0) > 0$. Then, at times sufficiently close to the starting time, the solution lies in the domain $q_1 > 0$; it will then be described by a system of ordinary differential equations (1.1), for which the properties of existence and uniqueness are known. The case $q_1(t_0) = 0$, $\dot{q}_1(t_0) < 0$ corresponds to impact: after the quantities $\dot{\mathbf{q}}(t_0 + \phi)$ have been calculated by using (2.1) and (2.4), it reduces to the previous case or (in plastic collision) to that considered below.

The greatest difficulties are encountered in analysing the solution under the conditions $q_1(t_0) = 0$, $\dot{q}_1(t_0) = 0$. We will prove the following proposition.

Theorem 1. If the quantity A_0 in (1.3) is negative at $t = t_0$, then a number $\tau > 0$ exists such that system (1.1), (1.2), (1.5) has a unique solution in the interval $(t_0, t_0 + \tau)$. For this solution the constraint remains active: $q_1(t) \equiv 0$.

Proof. Consider the function

$$L = ½/2q_1^2 - A_0 q_1 \tag{2.5}$$

By assumption, A_0 is negative in some neighbourhood of the initial point in the extended phase space. Consequently, the function L is non-negative and vanishes if and only if $q_1 = 0$, $q_1 = 0$. Because of inequality (2.3), this function does not increase on impact; in the domain $q_1 > 0$ its total derivative, evaluated along trajectories of (1.1), admits of the following estimate (using the inequality $L \geqslant 0$)

$$\frac{dL}{dt} = -q_1 \frac{dA_0}{dt} \leqslant -\frac{1}{A_0}\left|\frac{dA_0}{dt}\right| L \tag{2.6}$$

For sufficiently small values of τ, the right-hand side of (2.6) does not exceed C_1L, where C_1 is a constant. Hence it follows that

$$L(t) \leqslant L(t_0)\exp\{C_1(t - t_0)\}$$

and, due to the initial conditions, we conclude that $L(t) \equiv 0 \Rightarrow \mathbf{q}(t) \equiv 0$. This equality enables us to reduce (1.2) to a system of ordinary differential equations of order $n - 1$, and we conclude, on the basis of the general theorems, that the solution is unique.

The following example will show that the conditions of Theorem 1 are necessary.

Example. Consider the raising of a load on a lift. In a gravitational field, when the lift accelerates, the load maintains contact with the platform by Theorem 1. But if there are no external forces, solutions are also possible reflecting the repeated bouncing of a load with increasing amplitude when the lift acceleration increases. We will construct one such solution. To do this, we will first specify a (piecewise-linear) law $x(t)$ governing the motion of the load and then determine the dependence of the height of the platform on time, $S(t) \leqslant x(t)$, for which the load may move in accordance with that law.

Assuming that $t_0 = 0$, we construct a solution with impacts at times $t_k = t^{-k}$ and post-impact velocities $x(t_k + 0) = 4^{-k}$. In view of the absence of external forces the motion is uniform in intervals between impacts, and so

$$x(t_k) = \sum_{j=k}^{\infty} (t_j - t_{j+1})\dot{x}(t_{j+1} + 0) = \tfrac{1}{7} 8^{-k}, \quad k = 1$$

The law of motion of the platform for which the load moves in this way will be determined from the relations

$$S(t_k) = x(t_k), \quad \dot{x}(t_k + 0) - \dot{S}(t_k) = e(\dot{S}(t_k - \dot{x}(t_k - 0)) \tag{2.7}$$

where the coefficient of restitution $e \in (0, 1)$ is constant. It is not difficult to construct a twice differentiable function $S(t)$ satisfying all of conditions (2.7), whose second derivative is positive for $t > 0$ (there may be more than one such function). To do this, one can use one of the standard interpolation methods.

3. THE CASE OF SEVERAL IDEAL CONSTRAINTS

Theorem 1 can be extended to the case of a system with several ideal unilateral constraints. We will construct a local system of coordinates so that these constraints are expressed by the inequalities $q_i \geqslant 0$ $(i = 1, \ldots, k)$.

Free motion is described by Eqs (1.1). For the other type of motion—supported motion—these equations take a form analogous to (1.2): since the constraints are assumed to be ideal, we have

$$\dot{p}_i - \frac{\partial T}{\partial q_i} = Q_i + R^{(i)}, \quad \dot{p}_j - \frac{\partial T}{\partial q_j} = Q_j, \quad (i = 1, \ldots, k; \;\; j = k + 1, \ldots, n) \tag{3.1}$$

where $R^{(i)}$ is the reaction of $q_i \geqslant 0$. We eliminate the generalized accelerations $\ddot{q}_j$ from the second group of Eqs (3.1), and reduce the first group to the form

$$\mathbf{B}\ddot{\tilde{\mathbf{q}}} = \tilde{\mathbf{F}} + \tilde{\mathbf{R}}, \tag{3.2}$$

$$\tilde{\mathbf{q}} = (q_1, \ldots, q_k), \quad \tilde{\mathbf{F}} = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t) = (F_1, \ldots, F_k), \quad \tilde{\mathbf{R}} = (R^{(1)}, \ldots, R^{(k)})$$

where $\mathbf{B}$ is a symmetric positive-definite matrix of order k—the matrix of the first of the quadratic forms in the decomposition of the kinetic energy as a sum similar to (2.2)

$$T = T_0(\dot{q}_1, \ldots, \dot{q}_k) + T^*(p_{k+1}, \ldots, p_n) \tag{3.3}$$

Note that there are no products like $\dot{q}_i p_j$ on the right-hand side of formula (3.3). This is not because of some special form of the generalized coordinates, but due to the definition (1.1) of the generalized momenta p_j (see the lemma in [9]).

In order to determine the generalized accelerations and reactions of the constraints at time $t = t_0$ from Eqs (3.2), we use the complementarity conditions

$$\ddot{q}_i \geqslant 0, \quad R^{(i)} \geqslant 0, \quad \ddot{q}_i R^{(i)} \equiv 0 \quad (i = 1, \ldots, k) \tag{3.4}$$

As is well known [10], for any positive-definite matrix $\mathbf{B}$ and any matrix $\tilde{\mathbf{F}}$ the algebraic system (3.2), which is linear in $\ddot{\mathbf{q}}(t_0)$, $\tilde{\mathbf{R}}$, has a unique solution satisfying (3.4). Note that this in itself does not guarantee the existence of a continuous solution at $t > t_0$.

To describe the impacts at one or more unilateral constraints, we use (1.5). Note that in the general case the impact at several constraints is not well-defined, since the functions $I_i(\mathbf{q}, \mathbf{q})$ are discontinuous on the surfaces $q_i = 0$ [11]. Nevertheless, formulae (2.1) remain true for $j = k + 1, \ldots, n$ [12]. Consequently, the second term in formula (3.3) does not change at impacts, while the first does not increase. This is sufficient to prove the following proposition without specifying the laws of impact.

Theorem 2. If all the quantities F_i $(k = 1, \ldots, k)$ in formula (3.2) are negative at $t = t_0$, then a number $\tau > 0$ exists such that system (1.1), (3.1), (1.5) has a unique solution in the interval $(t_0, t_0 + \tau)$. The constraints remain active for this solution: $q_i(t) \equiv 0$ $(i = I, \ldots, k)$.

Proof. Consider the function

$$L = \tfrac{1}{2}(\mathbf{B}\dot{\tilde{\mathbf{q}}}, \dot{\tilde{\mathbf{q}}}) - (\tilde{\mathbf{F}}, \tilde{\mathbf{q}}) \tag{3.5}$$

This function is non-negative in some neighbourhood of the origin in the phase space and vanishes on the manifold $q_i = 0$, $\dot{q}_i = 0$ $(i = l, \ldots, k)$. Its derivative along trajectories of Eqs (3.1) admits of the following estimate

$$\dot{L} = \tfrac{1}{2}(\dot{\mathbf{B}}\dot{\tilde{\mathbf{q}}}, \dot{\tilde{\mathbf{q}}}) + (\mathbf{B}\ddot{\tilde{\mathbf{q}}}, \dot{\tilde{\mathbf{q}}}) - (\tilde{\mathbf{F}}, \dot{\tilde{\mathbf{q}}}) - (\dot{\tilde{\mathbf{F}}}, \tilde{\mathbf{q}}) =$$

$$= \tfrac{1}{2}(\dot{\mathbf{B}}\dot{\tilde{\mathbf{q}}}, \dot{\tilde{\mathbf{q}}}) + (\tilde{\mathbf{R}}, \dot{\tilde{\mathbf{q}}}) - (\dot{\tilde{\mathbf{F}}}, \tilde{\mathbf{q}}) \leqslant C_2 L, \quad C_2 = \text{const} \tag{3.6}$$

For impacts at one or several constraints, the first term in formula (3.5), as already noted, does not increase, while the second term may change owing to variation of one of the quantities F_i, if the corresponding constraint is relaxed at the time of impact. Let $h_s = |\, \dot{\mathbf{q}}(t_s + 0) - \dot{\mathbf{q}}(t_s - 0)\,|$ be the modulus of the velocity jump on impact at time $t = t_s \in (t_0, t_0 + \tau)$. Then the sum of the series with general term h_s does not exceed the variation of the vector-valued function $\mathbf{q}(t)$ in the interval $(t_0, t_0 + \tau)$, i.e. it is bounded (see the remark in Section 1).

Since $q_i = O(L)$, it follows that

$$L(t_s + 0) - L(t_s - 0) \leqslant C_3 h_s L(t_s - 0), \quad C_3 = \text{const} \tag{3.7}$$

Let (t_1, t_2) be some interval of impact-free motion. Then, by (3.6)

$$L(t_2 - 0) \leqslant L(t_1 + 0)\exp\{C_2(t_2 - t_1)\} \tag{3.8}$$

Now, using (3.7), we obtain

$$L(t_2 + 0) \leqslant L(t_2 - 0)(1 + C_3 h_2) \tag{3.9}$$

Combining inequalities (3.8) and (3.9) for the whole interval $(t_0, t_0 + \tau)$, we obtain

$$0 \leqslant L(t) \leqslant L(t_0)\exp(C_2\tau)\prod_s (1 + C_3 h_s) \tag{3.10}$$

That the infinite product in this formula is convergent follows from the above-mentioned convergence of the series with general term h_s. Since $L(t_0) = 0$, we conclude that $L(t) \equiv 0$, and that this equality remains valid as long as all the functions F_i in Eqs (3.2) are negative.

4. DISCONTINUOUS MOTIONS IN SYSTEMS WITH FRICTION

We will now discuss systems with dry friction, confining our attention to the case of one unilateral constraint $q_1 \geqslant 0$. The relation between the components of the reaction in Eqs (1.2) is described by the formulae

$$R_j = \varphi_j(\mathbf{q}, \dot{\mathbf{q}})R_1 \quad (j = 2, \ldots, n) \tag{4.1}$$

The first feature of systems with friction is the possibility that the quantity A_1 in Eq. (1.3) may be negative (the situations that arise when this happens are known as the Painlevé paradoxes). In addition, in impact with friction, inequality (2.3) does not follow directly from the laws of dynamics and has to be verified specially for the impact model actually adopted.

A direct extension of Theorem 1 is the following.

Theorem 3. If $A_0 < 0, A_1 > 0$ in formula (1.3) when $t = t_0$, and inequality (2.3) is satisfied for sufficiently small values of the approach velocity on impact against the constraint, then a number $\tau > 0$ exists such that system (1.1), (1.2), (1.4), (1.5) has a unique solution in the interval $(t_0, t_0 + \tau)$. The constraint remains active for this solution: $q_1(t) \equiv 0$.

The proof is identical to that of Theorem 1.

If one of the assumptions of Theorem 3 does not hold, the system may have a discontinuous solution. In the case $A_1 < 0, A_0 > 0$ (the non-uniqueness paradox), system (1.3) has two solutions compatible with (1.4): (1) $\ddot{q}_1 = A_0$, $R_1 = 0$ and (2) $q_1 = 0$, $R_1 = -A_0/A_1$. The first of these solutions corresponds to weakening of the unilateral constraint: $q_1 > 0$ when $t > t_0$. In the second solution the constraint remains active; here one obtains an infinite set of distinct solutions for $t > t_0$, since cessation of contact may occur at any of the times, as long as the non-uniqueness conditions are maintained. The "true" motion cannot be chosen out of all possible ones on the basis of the laws of dynamics; resolution of the paradox requires the adoption of additional physical assumptions.

For example, one can assume, following [13], that in the given situation rough bodies behave like smooth ones, one may modify the friction law [14], relax the requirement that the bodies in contact be absolutely rigid [14, 15] or choose a solution for stability [15]. All these methods lead to the first of the two solutions.

Note that, apart from the two continuous solutions in this case the system also admits of a discontinuous solution; substituting the initial data into the right-hand side of formula (1.5), we obtain a non-zero velocity jump. This possibility is analogous to that considered below.

In the case $A_1 < 0, A_0 < 0$ (the non-existence paradox), system (1.3), (1.4) is inconsistent. Consequently, the motion is discontinuous. Since the initial approach velocity of the bodies, $\dot{q}_1(t_0)$, is zero, this type of motion is known as tangential impact or impact-free collision [16]. On completion of the impact in the general case $\dot{q}_1(t_0 + 0) > 0$, and the constraint is weakened.

Let us assume now that the first two conditions of Theorem 3 are satisfied, i.e. $A_1 > 0, A_0 < 0$, but inequality (2.3) fails to hold for the given impact law (1.5) in as small a neighbourhood of zero as desired. There are several subcases:

(a) If $\lim q_1^+ > 0$ as $q_1^- \to -0$, the system experiences an impact analogous to tangential impact and the constraint is weakened.

(b) If $\lim q_1^+ = 0$ as $q_1^- \to -0$, and at the same time $\lim q_1^+/|q_1^-| = \theta > 1$, then, for the given initial data, the system with active constraint admits, besides the continuous solution, of discontinuous solutions. Such a motion includes an infinite number of impacts at times $t_k = \lambda\theta^{-k} + O(\theta^{-2k})$, where λ is an arbitrary positive number. Thus, we have here infinitely many possible discontinuous solutions.

(c) If $\lim q_1^+/|q_1^-| = 1$ as $q_1^- \to -0$, and the functions $I_i(\mathbf{q}, \mathbf{q}^-)$, describing the dependence of the momentum on the initial data of the impact are differentiable, then no discontinuous solution is possible. Since in that case inequality (2.3) may fail to hold, this statement may be viewed as a supplement to Theorem 3. It can be proved by the methods used above. Under our assumptions, the variation of the normal component of the velocity on impact is described by the formula

$$|\dot{q}_1^+| = |\dot{q}_1^-| + O\left(|\dot{q}_1^-|^2\right) \tag{4.2}$$

Estimate (2.6) remains valid for the function (2.5) on the flight sections, but at impacts against the constraint we have, by (4.2)

$$\Delta L = \Delta \dot{q}_1 O(L) \tag{4.3}$$

Formula (4.3) is analogous to (3.7), and therefore, reasoning by analogy with the theorem, we conclude that $L \equiv 0$.

Thus, discontinuous motions arise in system with friction in two cases: either $A_1 < 0$, or $\lim q_1^+/|q_1^-| > 1$. Note that the validity of these inequalities in actual systems depends on physical assumptions of different kinds: the function $A_1(\mathbf{q}, \mathbf{q})$ is uniquely defined by the law of friction (4.1), but additional hypotheses as to the laws governing the impacts are needed in order to determine the impact impulses in formula (1.5).

Below we consider an example of a system with Coulomb friction, using some of the best-known models of impact.

Example. An inhomogeneous disk on a rough support [14, 17]. Let G be the centre of mass of the disk, C its geometric centre, C' the point of contact, r the radius of the disk and let $a = |CG|$. The position of the disk in the vertical plane is defined by the coordinates of the point C in the system OXY (OX being the horizontal axis) and the angle ψ between CG and OX (see Fig. 1). The equations of motion under the action of the force of gravity and the reaction of the support may be formulated using the fundamental laws of dynamics

$$\begin{gathered} m(\dot{x} - a\dot{\psi}\sin\psi)^{\cdot} = R_X, \quad m(\dot{y} + a\dot{\psi}\cos\psi)^{\cdot} = -mg + R_Y \\ mk^2\ddot{\psi} = (r + a\sin\psi)R_X - a\cos\psi R_Y \end{gathered} \tag{4.4}$$

where m is the mass of the disk and k its central radius of inertia.

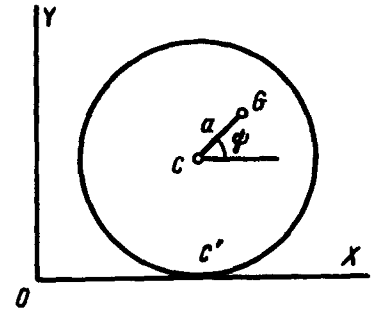

Fig. 1.

The unilateral constraint is represented by the inequality $y \geqslant r$; the components of the reaction are related by the equation $R_X = -\mu R_Y \mathrm{sign}(x = r\psi)$, where μ is the coefficient of sliding friction. We shall assume that at the starting time the normal component of the velocity of the point of contact equals zero, while the horizontal component is negative. Solution of system (4.4) leads to an equation of type (1.3) with

$$A_0 = a\dot{\psi}^2 \sin\psi - g,\quad A_1 = \frac{1}{mk^2}(k^2 + a^2\cos^2\psi - \mu a\cos\psi(r + a\sin\psi)) \tag{4.5}$$

Readers can convince themselves that the coefficients (4.5) may take values of both signs: A_0 is positive if the centre of mass lies below the geometric centre and the velocity of revolution is sufficiently high; A_1 is negative is the centre of mass is to the right of the geometric centre and the friction coefficient is sufficiently high, for example: $\psi = 0, \mu > (k^2 + a^2)/ar$.

The equations of the classical theory of impact may be obtained from (4.4) by replacing the derivatives of the velocities by increments and ignoring the effect of the gravitational force [12]

$$m\Delta(\dot{x} - a\dot{\psi}\sin\psi) = I_X,\quad m\Delta(\dot{y} + a\dot{\psi}\cos\psi) = I_Y$$
$$mk^2\Delta\dot{\psi} = (r + a\sin\psi)I_X - a\cos\psi I_Y,\quad \mathbf{I} = \int_{t_0}^{t_0+\Delta t}\mathbf{R}dt \tag{4.6}$$

where I_X and I_Y are the components of the impact impulse and Δt is the duration of the collision. It follows from Eqs (4.6) that

$$mV_X' = a_{11}I_X' + a_{12},\quad mV_Y' = a_{12}I_X' + a_{22}$$
$$a_{11} = 1 + (r + a\sin\psi)^2 k^{-2},\quad a_{12} = -a\cos\psi(r + a\sin\psi)k^{-2},\quad a_{22} = 1 + a^2k^{-2}\cos^2\psi$$

where $\| a_{ij} \|$ is a positive-definite matrix, the prime denotes differentiation with respect to the variable $\chi = I_Y$, which is monotone increasing in time and $V_X = \dot{x} + r\dot{\psi}$, $V_Y = \dot{y}$ are the velocity components of the point of contact C'.

We will first consider an impact law in the paradoxical case when $A_1 < 0$ at the initial instant. For sufficiently small values of χ, the quantity $\Delta\dot{y}$ will be negative. This indicates the existence of impact not only when there is an initial approach velocity ($\dot{y}(t)) < 0$), but also in the case discussed above, when $\dot{y}(t_0) = 0$ (tangential impact). The impact consists of several phases. In the first, the disk is "pressed" into the support: $V_Y' < 0$, and when that happens $mV_X' = \mu a_{11} + a_{12} > a_{12} - a_{11}a_{22}/a_{12} > 0$. After the relative slipping stops, the quantity V_X remains equal to zero, and then $mV_Y' = a_{22} - a_{12}^2/a_{11} > 0$, i.e. the vertical component of the velocity at the point of contact increases.

The time at which $V_Y = 0$ corresponds to the greatest deformation of the colliding bodies. If the impact is absolutely inelastic, it ends at this point. Elastic impact also includes a restoration phase. At the end of such an impact the disk receives a vertical velocity proportional to the coefficient of restitution of the impact impulse and the initial sliding velocity.

In the regular case $A_1 > 0$, one has $V_Y' > 0$ throughout the impact. If the initial sliding velocity is not zero, it does not change direction on impact, provided that the vertical component of the relative velocity is sufficiently small.

Use of Newton's kinematic coefficient of restitution, Poisson's dynamic coefficient or the Boulanger–Strong energy coefficient leads to the same computational results (see [18]): $\dot{q}_1^+ = e\dot{q}_1^-$. Here, according to Theorem 3, we conclude that with the given initial data the system admits of no discontinuous motions. With the coefficient of restitution proposed in [18], one reaches a different conclusion: apart from continuous motion, the system will also admit of impact (subcase a, considered above).

We now proceed to impact models based on allowance for deformations in the contact area. Without increasing the number of dimensions of the system, we will assume that the inequality $q_1 < 0$ is possible on impact, in which case the reaction is given as a certain explicit function $\mathbf{R}(\mathbf{q}, \mathbf{q})$, different from zero only in the domain $q_1 < 0$. Retaining a Coulomb law of type (4.1) for the components of the reaction, we obtain a family of models, each of which is defined by a function $\mathbf{R}_1(\mathbf{q}, \mathbf{q})$ and described by a system of ordinary differential equations. In particular, Eq. (1.3) takes the following form in the domain $q_1 < 0$

$$\ddot{q}_1 = A_0 + A_1 R_1(\mathbf{q}, \dot{\mathbf{q}}) \tag{4.7}$$

Notwithstanding the quantitative difference between different models (4.7), all such models share certain qualitative properties.

First, tangential impact occurs only when no continuous solutions exist ($A_0 < 0, A_1 < 0$). In the case of non-uniqueness ($A_0 > 0, A_1 < 0$ for $t > t_0$), the constraint is weakened after action of a "finite" force A_0. Consequently, the paradox is resolved in favour of continuous motion. In this respect these models depart from the classical model.

Second, if no tangential impact exists, sliding stops and the disks separates from the support.

Third, in the regular case, for sufficiently small values of the initial approach velocity $|\dot{q}_1|$, the direction of sliding does not change during impact; by energy arguments, this implies that $q_1^+ \leqslant |q_1^-|$. Consequently, the conditions of Theorem 3 are satisfied.

There are also other models of rough body impact, in which the friction does not obey Coulomb's law (see, e.g. [19]). In that case the third condition of Theorem 3 may fail to hold, leading to the existence of discontinuous solutions besides continuous ones.

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